Kant und die Berliner Aufklärung

Akten des
IX. Internationalen
Kant-Kongresses

Band II: Sektionen I–V

Herausgegeben
im Auftrag der Kant-Gesellschaft e.V.

von
Volker Gerhardt, Rolf-Peter Horstmann
und Ralph Schumacher

Walter de Gruyter · Berlin · New York
2001
Incongruent Counterparts and the Nature of Space: Demystifying their Reappearance in Kant's Writings

Prasanta S. Bandyopadhyay, Bozeman

Overview

Two objects are called incongruent counterparts if and only if one object is a mirror image of the other, and yet one could not occupy the region of space just vacated by the other. (For a sophisticated discussion on this and other related notions of incongruent counterparts, see Earman 1991) In his pre-critical writings, especially in the “Regions in Space” (1768), Kant uses the existence of incongruent counterparts to argue that space is absolute, i.e., not relational, and intuitive, i.e., not a concept. By “absolute space”, Kant means space that exists independent of matter. In the critical period of his career, incongruent-counterparts reappear in the Prolegomena (1783, § 13). Commentators like Beck contend that Kant employs incongruent counterparts in 1783 to reinforce the thesis that space is both absolute and intuitive. Here, Kant takes “absolute space” to be one which exists over and above a system of relations among objects. That is to say, space is not reducible to a system of relations.

Two questions then arise. First, why does Kant recycle incongruent counterparts to argue the same point that space is absolute and intuitive and second, why does Kant not use them in the First Critique (1781), to argue the same point about space? Commentators like Meerbote disagree with the significance of these questions. Meerbote contends that incongruent-counterparts play different roles in the “Regions in Space” and the Prolegomena. In the “Regions in Space”, Kant argues that space is both absolute and intuitive, whereas, it is alleged, in the Prolegomena, space is both relational and intuitive. I call this latter account the relational account of space. I contend that the relational account fails to provide a unified approach to Kant's philosophy of geometry since it is incompatible with Kant's anti-reductionism research program in philosophy of mathematics. To respond to the first question, “Why does Kant recycle incongruent counterparts in his writings?”, I argue that Kant's notion of proof underwent a change over a period of fifteen years (1768-1783). I think that a satisfactory answer to this question will also provide a satisfactory answer to the second question: Why does Kant not use them in the First Critique (1781), to argue the same point about space? First, I interpret his 1768 argument for absolute space as a reductio ad absurdum of the relational theory. Second, I take his 1783 argument for space as a direct proof for space being absolute. I attribute the change in his style of proof to the “transcendental philosophy”, which is well developed at the time of the First Critique (1781). This change prompts him to use incongruent counterparts for the first time in the critical period of his career. My account reinstates Beck's interpretation about the absolute character of space, thus providing both a historically interesting and a philosophically credible account of Kant's motivation for recycling incongruent counterparts.

A Reconstruction of Beck's Interpretations

“Regions in Space”

Kant asks us to consider a thought-experiment. Suppose God created only one solitary hand and nothing else in the universe. If one is a relationalist, then one will argue that there exists no other object with respect to which one can tell precisely that it is a left or a right hand. Hence, on the relationalist reading, the exact nature of the hand is indeterminate. Kant contends, however, that all references including subsequent quotations of the Critique of Pure Reason are from the translation by P. Guyer and A. Wood, Cambridge University Press, 1998.


This catchy expression, “anti-reductionism” is due to Brittan. See Brittan, Theory of Science, Princeton University Press, 1979. However, I don’t know whether Brittan will agree with my reading of Kant's anti-reductionism here.

There is a further reason for arguing that Kant cannot be a relationalist during the period of the Prolegomena. See footnote 19.
though there is only a solitary hand, it must be either left or right. If the hand, he rejoins, is necessarily either left or right, then that hand must be left or right with respect to a thing. Since in the universe there is nothing except the solitary hand, the hand must be left or right with respect to a space that exists over and above the solitary hand. Hence, it follows that there is absolute space over and above the existing objects that are embedded in that space.

Prolegomena

Kant dramatizes the puzzle of incongruent counterparts with a pair of gloves. One is a left glove and the other one is a right glove. Both look alike and have the same relation to their corresponding parts. For example, the angle between the thumb and the index finger is one and the same for both the gloves, the length of each index finger is also the same and so on. However, one cannot be replaced by the other, since they can’t occupy the same space. What Kant intends to say is that there is no rigid motion of the left glove that can superimpose it into the right glove while the left glove shares the same surface with the latter. Although from the perspective of relations between the two gloves, they are equal and the same, we can differentiate one from the other. How is this possible? Kant writes, “Here, then, is an internal difference between the two [gloves], which difference our understanding cannot describe as internal and which only manifests itself by external relations in space.”

The question may arise why does the external relation to space show that space is absolute rather relational? The reason for Kant’s considering space as absolute is that the gloves’ external relation to space is in fact an implication of their relation to the whole space. Kant writes, “Space is the form of the external intuition of this sensibility, and the internal determination of every space is possible only by the determination of its external relation to the whole space of which it is a part ...”

Here, by “the whole of space”, Kant means that it is the nature of space as a whole that determines the handedness of the gloves rather than the existing relation of the gloves to other objects that determines their handedness. Only philosophers who subscribe to the absolute theory of space can accept the idea of space as a whole determining the direction of a hand. So

11 A more careful formulation of this point can be found in R. Frederick’s article, “Introduction to the Argument of 1768”. Here, he writes, “[hence, the hand must be left or right at least partly in virtue of its relation to absolute space”. Emphasis is mine. This passage occurs in J Van Cleve and R. Frederick (Eds.) The Philosophy of Right and Left, Kluwer, Academic Publishers, Dordrecht, 1991.


13 Prolegomena § 13 (see footnote 12)


I take Kant to be saying that space is absolute, not a system of relations among objects.15 This is also Beck’s reading of Kant’s argument. However, this raises the question, “why does Kant recycle incongruent counterparts at least twice to argue that space is both absolute and intuitive”? Interestingly, the relationist account provides an answer, however wrong, to the question.

The Relationist Contention

Commentators like Meerbote disagree with the above interpretation of Kant’s 1783 argument. Referring to the last but one passage I have just quoted from the Prolegomena, he contends that Kant explains the inner difference between incongruent counterparts in terms of a difference determinable by means of non-discursive external spatial relations. And this spatial difference, according to Meerbote, can be expressed relationally without invoking absolute space. He concludes “that Kant’s resolution consists of saying that there is no non-relational difference in the case of incongruent-counterparts, and that the difference there must be relational and intuitive”.16 Contrasting Kant’s position discussed in the 1768 paper with this change, Meerbote says “[on] this reading, Kant, who earlier in his thinking on these matters had argued that left-handedness, for example, must be understood as a non-relational spatial property, is now agreeing with Leibniz that it is relationally determinable”.17 So, Meerbote concludes that Kant’s 1783 argument is evidence for Kant’s agreement with Leibniz regarding space being relational.

What is Wrong with the Relationist Account?

I argue that Meerbote’s account overlooks a deep philosophical difference between the two research programs, Kant’s and Leibniz’s. Kant’s research program in philosophy of mathematics is entirely anti-reductionist, whereas Leibniz’s research program is thoroughly reductionist. According to Kant, all mathematical propositions are synthetic (Critique, B 14). Kant admits that although in a mathematical deduction, one step follows from another analytically, that is, in accordance to the law of contradiction, the premises to begin with and the conclusion we end up with in the deduction are synthetic. Leibniz contends, to the contrary, that all mathematical propositions are


17 Ibid.
analytic. On Leibniz’s view, all mathematical propositions can be reduced to definitions and the principle of contradiction. By “mathematical propositions”, both Leibniz and Kant include algebraic, geometric and arithmetical propositions.

I argue that when Kant holds that mathematical propositions without any exception are synthetic he has in mind how objects referred to in a theory or in a sentence are capable of being intuited in an objective fashion in a spatio-temporal framework. Kant thinks that mathematical propositions involve synthesis, which, in turn, involves intuition. What he means by intuition is that the object of our discourse or objects referred to in a theory are capable of being experienced by human beings. Objects of discourse must be objects that are knowable. However, according to him, objects that are knowable are those that are capable of occupying determinate space-time positions. Objects that are capable of occupying determinate spatio-temporal position are objects of our sensible intuition. He distinguishes this type of object from the other kinds like “God” and “soul” which are not subject to our possible sensible intuition.

Sensible objects have objective reality that presuppose space and time as a priori conditions of experience. Unless space and time exist over and above the system of relations among objects, they cannot be a priori conditions of our experience. On Kant’s account, systems of relations are not given a priori since they depend on objects and their existing relations. If space and time were a system of relations among objects, they would be bound to change due to corresponding changes in the system of relations, thus failing to be a priori conditions of our experience.

Kant contends that the real difference between incongruent counterparts is manifested to us because it is with respect to the absolute “objective” space that we are able to distinguish the left glove from the right glove. According to him, it is the nature of space that contributes to the intrinsic difference between incongruent counterparts. This is what Kant means when he says that the determination of the difference between incongruent-counterparts can be understood only with respect to their external relations to the “whole of space”. Here, the whole of space is not something that is reducible to its parts or to a system of relations among objects. Rather its parts or the existing system of relations is possible because of the existence of the whole of space. In short, for Kant, space is not something that is reducible to a system of relations among objects, whereas, for Leibniz, space is nothing other than a system of relations among objects. Hence, while Kant is an anti-reductionist in mathematics, Leibniz is a reductionist.

In this section, I have argued that the relationalist reading of the Prolegomena is mistaken. If the relational account is wrong, then it leads us back to the question I have raised before, viz, why does Kant recycle incongruent-counterparts to argue the same point that space is both absolute and intuitive in at least two occasions without being repetitious? An answer to this question lies in Kant’s changing views on the notions of proof.

Changes in Kant’s Notions of Proof

Recall Kant’s thought-experiment of the existence of a solitary hand. Kant contends that if the relational theory were true, then we had to conclude that the hand in question was indeterminate. We have seen before that Kant does not think that the hand is indeterminate since if it were the only one hand then

space and time? Are they actual entities? Are they only determinations or relations of things . . . ? (Critique B 38). Kant thinks that space is ontologically prior to its objects and is not reducible to a system of relations among objects, possible or actual. I think that the question of the metaphysical investigation about the nature of space leads to his epistemological investigation. So, it will be a mistake to identify Kant as a philosopher whose only query with respect to space is metaphysical. If we continue to follow what he has said in the Critique B 38, then we will see that clearly: "Are they [i.e., space and time] only determinations or relations of things, yet ones that would pertain to them even if they were not intuited?" In The Metaphysical Exposition of Space, Kant raises an epistemological point regarding our ability to represent an empty space (i.e., a space without any object in it) and at the same time our inability to represent an object that does not occupy a space. Although, Kant raises epistemological questions regarding the nature of space, he raises them as a follow-up of his metaphysical questions. This is why, I call Kant’s interest primarily quasi-metaphysical. Leibniz, on the other hand, is primarily keen on the epistemological issue of spatial and temporal relations among objects. He provides an answer to the question, "how is our knowledge of space possible?" In his Fifth Letter to Clarke, Leibniz writes: "I will here show how men come to form themselves the notion of space. They consider that many things exist at once and they observe in them a certain order of co-existence, according to which the relation of one thing to another is more or less simple. This order, is their situation or distance. When it happens that one of those co-existent things changes its relation to a multitude of others, which do not change their relation among themselves; and that another thing, newly come, acquires the same relation to the others, as the former had; then we say, it is come into the place of the former; and this change, we call a motion in that body, wherein is the immediate cause of the change. ... And supposing or leaguing, that among those co-existents, there is a sufficient number of them, which have undergone no change; then we may say, that those which have such a relation to those fixed existents, as others had to them before, have now the same place which others had. And that which comprehends all those places is called space." Leibniz argues and I think rightly so that we cannot have knowledge of space without knowing the existence of some object, possible or actual, occupying the space. Recall Kant’s thought-experiment about the solitary hand and see how Leibniz would have responded to it. On Leibniz’s reading, one cannot determine or know whether the hand is right or left since it is not related to an object with respect to which we can determine whether it is a left or a right hand. Hence, on this reading, the solitary hand is indeterminate. Since according to Leibniz, we cannot determine or know whether the hand is left or right, I call Leibniz’s interest in matters related to the notion of space as primarily epistemological. However, Leibniz is also interested in the ontological question regarding space. I argue that his ontological investigation is an off-shot of his epistemological investigation. Since, according to the relational account, we cannot know whether the solitary hand is left or right, we can make a plausible metaphysical claim based on this epistemological argument that space cannot be ontologically prior to objects. If there is something called a space, then, on the relationalist reading, it must be a "fiction" built from a system of relations among objects. So with regard to space, Leibniz’s interest is quasi-epistemological, while Kant’s interest is quasi-metaphysical.
of how to find it. Here is an example of an existence proof. There is a well-known problem in elementary mathematics whether we can prove that in any sufficiently large city at least two people must have exactly the same number of hairs on their heads. In the case of a big city like New York City all one need to know is that the number of hairs on any given head is less than the city’s population of roughly 10,000,000. If each person is tagged by his or her specific number of hairs, at least two people must be tagged by the same number. That is, two people must have the identical number of hairs.

One proof of this problem adopts the form of an existence proof. Consider, for example, the residents of New York City. Put persons with no hairs on their heads in group zero, one person on their heads in group one, those with two hairs in group two, and so on. By hypothesis, we will need at most ten million groups to accommodate everyone. Start with the person with the fewest hairs on his or her head. Check to see if there is another person with the same number. If so, our proof is complete. If not, move on to the person with the next fewest hairs on his or her head and repeat the process. We have more people than hairs. So we will have to find a match before we get to the last person of the population. QED.

Examples of existence proof can be also found in geometry. Suppose I want to prove that if a triangle is not isosceles, then the bisector of one angle must meet the perpendicular bisector of the opposite side in a point outside the triangle. The proof shows that if the triangles are not isosceles then there is a point that lies outside the triangle. The proof does not pinpoint the exact location of that point outside the triangle. On my proof, I have to show only that such a point exists outside the triangle. This kind of proof is known as an existence proof. Sometimes mathematicians are not satisfied with this existence proof. They offer a constructive proof that will show, if possible, where exactly, that particular point lies outside the triangle. Recall the problem where we are confronted to prove that there are at least two people with the same number of hairs on their heads. In this scenario, mathematicians interested in a constructive proof want to know, if possible, exactly which two people have the same number of hairs. So there are two senses of a direct proof, an existence proof and a constructive proof.

There are two distinct senses of a direct proof in mathematics. The first is called an existence proof and the second is called a constructive proof. An existence proof establishes the existence of some entity without informing us of how to find it. Here is an example of an existence proof. There is a well-known problem in elementary mathematics whether we can prove that in any sufficiently large city at least two people must have exactly the same number of hairs on their heads. In the case of a big city like New York City all one need to know is that the number of hairs on any given head is less than the city’s population of roughly 10,000,000. If each person is tagged by his or her specific number of hairs, at least two people must be tagged by the same number. That is, two people must have the identical number of hairs.

One proof of this problem adopts the form of an existence proof. Consider, for example, the residents of New York City. Put persons with no hairs on their heads in group zero, one person on their heads in group one, those with two hairs in group two, and so on. By hypothesis, we will need at most ten million groups to accommodate everyone. Start with the person with the fewest hairs on his or her head. Check to see if there is another person with the same number. If so, our proof is complete. If not, move on to the person with the next fewest hairs on his or her head and repeat the process. We have more people than hairs. So we will have to find a match before we get to the last person of the population. QED.

Examples of existence proof can be also found in geometry. Suppose I want to prove that if a triangle is not isosceles, then the bisector of one angle must meet the perpendicular bisector of the opposite side in a point outside the triangle. The proof shows that if the triangles are not isosceles then there is a point that lies outside the triangle. The proof does not pinpoint the exact location of that point outside the triangle. On my proof, I have to show only that such a point exists outside the triangle. This kind of proof is known as an existence proof. Sometimes mathematicians are not satisfied with this existence proof. They offer a constructive proof that will show, if possible, where exactly, that particular point lies outside the triangle. Recall the problem where we are confronted to prove that there are at least two people with the same number of hairs on their heads. In this scenario, mathematicians interested in a constructive proof want to know, if possible, exactly which two people have the same number of hairs. So there are two senses of a direct proof, an existence proof and a constructive proof.

**What Does Kant mean by a Direct Proof?**

When Kant addresses the possibility of mathematical knowledge, he uses both notions of direct proof. Mathematical knowledge for Kant is the knowledge
gained by reason from the *construction* of concepts (Critique, A 713/B 741). What is it to construct a concept? Kant writes, “[t]o construct a concept means to exhibit *a priori* the intuition which corresponds to the concept.” For Kant, an intuition is the direct apprehension of an individual object. To provide an existence proof for an object, it is necessary to be able to specify that object. According to Kant, this object must be specified by ostension that is also known as intuition.

Ostension or intuition plays a crucial role in Kant’s philosophy. He writes, (Critique, B 154) that the possibility of construction insures the “existence” of the object in question. It is important to note that both geometry and arithmetic are “ostensive”24. He discusses at the passage (B 15-16) how numbers can be ostended by fingers, strokes on a page, and how all of them are spatial representations. In constructing a geometrical figure, a triangle for example, we often represent it by a figure drawn on a blackboard. In the same way we “construct” arithmetical or algebraic concepts when we represent the individual quantities, perhaps by the fingers of a hand, perhaps by numerals or letters.

It is unclear whether Kant is aware of these distinct senses of direct proofs. When he writes, “[t]o construct a concept means to exhibit *a priori* the intuition which corresponds to the concept”, he may mean by “constructing a concept”, either an “existence proof” or a “constructive proof”. Although it is unclear whether Kant knows the distinction, it is evident historically that he is familiar with both indirect and direct kinds of proofs used in mathematics. Further, it is evident from his writings that due to his “transcendental method” he is more favorably disposed to the use of direct proofs rather than indirect proofs after the publication of the First Critique (1781). He takes reductio proof as “a last resort” (Critique, B 818).25 Here, objects to be proved fail to be ostended, and therefore they are not possible objects of intuition. It was no wonder that the argument Kant employed in the Prolegomena (1783 § 13) was an example of a direct proof: To make his transcendental philosophy consistent with his earlier works, he recycled incongruent-counterparts in the critical period of his career to argue that space is both absolute and intuitive.

So far what I have argued in the last three sections can be summarized as follows: (i) Kant’s notions of proof undergo a change over a period of fifteen years. We are able to read this change in the First Critique when we come across his unfavorable observations toward indirect proofs as opposed to direct proofs. (ii) There are two senses of a direct proof, (a) an existence proof, and (b) a constructive proof. Finally and (iii), although it is unclear whether Kant is aware of this distinction, it is quite clear that he is favorably disposed to the use of direct proofs.

Why my Interpretation is Better

In the literature on scientific explanation,26 an explanation is considered to be a good scientific explanation if it is able to unify diverse phenomena. A theory that has the ability to provide a unifying explanation for several apparently unrelated phenomena is hailed as a good theory. For example, we prefer Einstein’s theory of relativity to Newton’s theory because of this reason. Using Einstein’s theory of relativity, we can explain (i) the occurrence of a red-shift, (ii) the bending of light in front of massive nearby objects and finally (iii) the precession rate of Mercury’s perihelion with sufficiently precise details. Although the ability to unify diverse items under one banner is canvassed as a plus point for scientific explanation, I think that this ability should also be counted as an added advantage for philosophical accounts that provide a unifying view of several apparently disjoint subjects. I claim that my interpretation has the ability to unify different parts of Kant’s views on mathematics. There are two independent considerations for my claim.

(i) Kant takes geometry to be an intuitive enterprise. By this he means that a geometric proof depends on constructing a figure in intuition. In geometry, on the other hand, to show that a proof is impossible, we could often assume to the contrary that the alleged proof is possible. Thus, we end up deriving an inconsistent result from this assumption. An indirect proof depends on constructing a figure in intuition, which turns out not to be a possible object of intuition. How could we resolve this puzzle that arises in Kant’s transcendental philosophy? One solution to this puzzle is to reject an indirect proof as a valid proof in Kant’s system, because we cannot construct a figure that is not a possible object of intuition. This makes Kant’s philosophy of mathematics coherent by making his pre-critical writings consistent with the Prolegomena.

(ii) It is a well-known metalogical result that if one accepts the law of bivalence, then one is committed to accepting an indirect proof as a valid method of proof. According to the law of bivalence, every proposition must be either true or false. I argued that Kant is skeptical about reductive proofs. For an object to be a possible object of intuition it must be ostended. He argues that for an object to be ostended, it must satisfy two requirements; the


25 When I mentioned this passage to Friedman, he told me that it seemed like Kant considered a reductio proof as “an invalid argument.” (In a private conversation, APA, Pacific Divisional Meeting, 1996). I don’t think that for Kant a reductio proof is an analogue of an invalid argument. I have given a justification for Kant’s thought in next section.

requirement of logical possibility and that of real possibility. It is logically possible to imagine that two straight lines can enclose a space. Our imagination satisfies the first requirement, since we do not commit a self-contradiction if we try to imagine so. The former fails to be ostended, because it does not satisfy the second requirement. In a reductio proof, however, neither of the requirements is satisfied. The object to be proved does not get ostended. Therefore, it cannot be a possible object of intuition. This could be a reason why Kant recommends that a reductio proof should be a last resort. It further follows from this that the law of bivalence must be called into question if we use modus tollens with the above metalogical result. This, in turn, supports Brittan’s contention that Kant dispenses with the law of bivalence.

Summing Up

In the pre-critical period especially when Kant wrote the “Regions in Space”, and in the critical period, especially when he wrote the Prolegomena, Kant used incongruent-counterparts to argue both that space is absolute and intuitive. I asked a question, “why did he recycle incongruent counterparts to establish the same points about space twice?” Commentators like Meerbote argued that Kant’s argument in the Prolegomena was intended to establish that space is both relational and intuitive. I argued that this relational reading did injustice to Kant’s anti-reductionism in philosophy of mathematics. I argued, to the contrary, that Kant provided in 1768 an indirect proof for absolute theory and he provided in 1783 a direct proof for absolute theory. This change in his concept of proof was due to the emergence of the transcendental philosophy in which his dissatisfaction with indirect proofs was evident. Why does Kant not use incongruent counterparts in the First Critique (1781)? Here is the answer. The most relevant aspect of my interpretation rests on exploiting a section of the Critique called “Transcendental Doctrine of Method”. In this section, Kant commented on both philosophical and mathematical methods. This was written in 1781 and remained unchanged when the Second Critique (1787) was published. After the publication of the First Critique, Kant’s immediate major work was the Prolegomena in which Kant first got an opportunity to address incongruent counterparts. And Kant took that opportunity.

I conclude that my account both demystifies Kant’s reasons for recycling incongruent counterparts in the Prolegomena and provides a unified approach to his philosophy of geometry. Interestingly, my account further defends Beck’s interpretation that Kant uses incongruent counterparts in the

Prolegomena to argue that space is both absolute and intuitive with a rationale for Kant’s use of incongruent counterparts, that is missing in Beck’s writings.

---

28 See Brittan (this year Kant Congress), “Transcendental Idealism, Empirical Realism, and the Completeness Principle”. See also his Kant’s Theory of Science.

29 The author wishes to thank the late L. Beck, J. Allard, G. Brittan Jr., M. Curd, L. Falkeinstein, M. Friedman, J. Earman, R. Meerbote, E. Watkins and P. Weirich for both suggestion and encouragement. The author is especially indebted to Professor Beck for allowing him to present a version of this paper in Beck’s summer seminar at the University of Rochester in 1996. He is also thankful to William Donaldson, Michael Mathias, and Phil Mousch for helpful comments on an earlier version of the paper.